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The Mathematics Educator
2011/2012 Vol. 21, No. 2, 11–33

Moving Toward More Authentic Proof Practices in Geometry

Michelle Cirillo and Patricio G. Herbst

Various stakeholders in mathematics education have called for increasing the role of reasoning and proving in the school mathematics curriculum. There is some evidence that these recommendations have been taken seriously by mathematics educators and textbook developers. However, if we are truly to realize this goal, we must pose problems to students that allow them to play a greater role in proving. We offer nine such problems and discuss how using multiple proof representations moves us toward more authentic proof practices in geometry.

Over the past few decades, proof has been given increased attention in many countries around the world (see, e.g., Knipping, 2004). This is primarily because proof is considered the basis of mathematical understanding and is essential for developing, establishing, and communicating mathematical knowledge (Hanna & Jahnke, 1996; Kitcher, 1984; Polya, 1981; Stylianides, 2007). More specifically, in describing proof as the "guts of mathematics," Wu (1996, p. 222) argued that anyone who wants to know what mathematics is about must learn how to write, or at least understand, a proof. This comment complements the call to bring students' experiences in school mathematics closer to the discipline of mathematics, that is, the practices of mathematicians (Ball, 1993; Lampert, 1992; National Council of Teachers of Mathematics [NCTM]

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2000). This idea is not new: A number of curriculum theorists from Dewey (1902) to Schwab (1978) have argued that the disciplines should play a critical role in the school curricula. Thus, by engaging students in *authentic mathematics*, where they are given opportunities to refute and prove conjectures (Lakatos, 1976; Lampert, 1992; NCTM, 2000), teachers can create small, genuine mathematical communities in their classrooms (Brousseau, 1997).

Through the introduction of the *Standards* documents (1989, 2000), NCTM put forth some significant recommendations related to the *Reasoning & Proof* and *Geometry* standards that have had the potential to impact the high school geometry curriculum. First, it has been recommended that reasoning and proof should not be taught solely in the geometry course, as it typically has been done in the United States. Rather, instructional programs in all grade bands

- should enable students to recognize reasoning and proof as fundamental aspects of mathematics;
- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs; and
- select and use various types of reasoning and methods of proof. (NCTM, 2000, p. 56)

Other calls to increase attention to reasoning and proof come from descriptions of mathematical proficiency. For example, the National Research Council (2001) recommended that students develop the capacity to think logically, to justify, and, ultimately, to prove the correctness of mathematical procedures or assertions (i.e., adaptive reasoning). More recently, the U.S. Common Core State Standards document (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) included, as one of their Standards for Mathematical Practice, the ability to construct viable arguments and critique the reasoning of others.

Despite these recommendations, in the United States the high school geometry course continues to be the dominant place where formal reasoning and the deductive method are

learned (Brumfiel, 1973; Driscoll, 2011; Yackel & Hanna, 2003). One reason for this is practical: After students conjecture about the characteristics and relationships of geometric shapes and structures found in the real world, geometry offers a natural space for the development of reasoning and justification skills (NCTM, 2000). However, even in the high school geometry course, students are typically not provided the kinds of experiences recommended in the standards documents. For example, in her study on teachers' thinking about students' thinking in geometry, Lampert (1993) outlined what doing a proof in high school geometry typically entails. According to Lampert, students are first asked to memorize definitions and learn the labeling conventions before they can progress to the reasoning process. They are also taught how to generate a geometrical argument in the two-column form where the theorem to be proved is written as an *if-then* statement. After students write down the "givens" and determine what it is that they are to prove, they write the lists of statements and reasons to make up the body of the proof. In this context, there is never any doubt that what needs to be proved can be proved, and because teachers rarely ask students to write a proof on a test that they have not seen before, students are not expected to do much in the way of independent reasoning. Similarly, through their analyses, Herbst and Brach (2006) argued that the norms of the situation of doing proofs do not necessarily support students through the creative reasoning process needed to come up with arguments on their own.

Another recommendation that has had the potential to impact the high school geometry curriculum is related to the modes of representation that are used to communicate mathematical proof. In the 1989 NCTM Geometry Standard, two-column proofs (which have typically been the proof form presented in U.S. textbooks) were put on the list of geometry topics that should receive "decreased attention" (p. 127). In the 2000 *Standards,* NCTM clarified its position, stating, "The focus should be on producing logical arguments and presenting them effectively with careful explanation of the reasoning rather than on the form of proof used (e.g., paragraph proof or two-column proof)" (p. 310). In other words, it is the argument, not the form of the argument, that is important.

Since these recommendations have been published, we have begun to see some changes to the written curriculum (i.e., textbooks). For example, many authors have addressed the proof form recommendation by promoting paragraph and flow proofs in their textbooks (see, e.g., Larson, Boswell, & Stiff, 2001). *Discovering Geometry* (Serra, 2008) is another example of a curricular shift in which the author expanded the role of the students by asking them to discover and conjecture through investigations but delays the introduction of formal proofs until the final chapter of the textbook. Most recently, the CME (Center for Mathematics Education) Project's *Geometry* (Education Development Center [EDC], 2009) asks students to conjecture and analyze arguments, proposes a variety of ways to write and present proofs, and asks students to identify the hypotheses and conclusions of given statements.

While we do not necessarily endorse all of these changes, we see these curricular adjustments as evidence that mathematics educators and textbook developers are, in fact, rethinking the geometry course. Through our research, however, we have noticed that even when teachers share this goal, many find it difficult to move away from the two-column proof form where students are provided with "givens" and a statement to prove (Cirillo, 2008; Herbst, 2002). In fact, the two-column form is so prominent that some research has shown that when proofs are written in other forms (e.g., paragraphs), high school students are unsure of their validity (McCrone & Martin, 2009).

One reason that the two-column form continues to dominate geometry proof is likely related to the "apprenticeship of observation" (Lortie, 1975) where teachers tend to teach in ways that are similar to how they were taught as students. We argue that this version of "doing proofs" does not do enough to involve students in the manifold aspects of proving that are found in the discipline of mathematics. This is important because, unless we expand our vision of proving in school mathematics, we cannot fully realize the aforementioned goals for mathematical proficiency and of NCTM's *Reasoning & Proof* and *Geometry* Standards. The focus of this article is on NCTM's recommendations for students to make and investigate conjectures, develop and evaluate mathematical proofs, and select and use various types of reasoning and methods of proof. Through our examples, we focus on the recommendation to expand the role of the student in the work of developing proofs and support this work through the selection of various proof representations.

In this paper, we first provide some historical context that sheds light on the prominent position that the two-column proof form holds in the geometry course. We do this in order to show how the student's role in proving has been narrowed over time. We then present a set of problems that are intended to expand the role of students by providing them with opportunities to make and investigate conjectures and develop and evaluate mathematical proofs. Finally, we discuss various proof forms as *representations* used to communicate mathematics. We conclude with a brief discussion of how these activities allow students to participate in more authentic proof practices in geometry.

Historical Context

A second reason that the two-column proof holds such a prominent position in the geometry course is historical. A perusal of American geometry textbooks covering the last 150 years reveals that problems where students are expected to produce a proof have changed considerably. As Herbst (2002) noted, the custom of using a two-column proof developed gradually. Before the 20th century, students were expected to prove statements in which geometric objects are referred to by their general names (e.g., triangle, angle) rather than by the labels for specific objects (e.g., $\triangle ABC$, $\angle ABC$). Students also had the chance to use deductive reasoning to determine the claim of their proof. For example, in response to a question about a generally described figure, they might be expected to develop a conjecture and prove it. Although less common, some problems (those problems left for independent exploration) included finding the conditions or hypotheses (i.e., the "givens") on which basis one could claim a certain conclusion.

During the 20th century, the student's role in proving substantially narrowed. It is interesting that this narrowing occurred simultaneously with the standardization of the two-

column form for writing proofs. If a goal for our students is simply to use the "givens" to construct the statements and the reasons that prove a conclusion, then the two-column form offers a useful scaffold to assist students in this work. Were we to increase the share of labor that students do when proving, however, we might have to think of other types of problems and forms of representation to support and scaffold their work. In thinking about expanding the student's role in the proof process, two questions are important to consider: What kinds of problems might be posed to increase students' share of the labor? What kinds of support, other than the traditional two-column scaffold, could be provided to students to do this work? We address these two questions in the sections that follow.

Expanding the Role of the Student Through Alternative Problems

One reason that the two-column form has come under so much scrutiny in recent times is related to the belief that it is not an *authentic* form of mathematics. For example, in *A Mathematician's Lament*, after presenting a two-column proof (that demonstrates that an angle inscribed in a semicircle where the vertex is on the circle is a right angle), Lockhart (2009) stated, "No mathematician works this way. No mathematician has ever worked this way. This is a complete and utter misunderstanding of the mathematical enterprise" (pp. 76–77). A critical piece that has been lost in our modern version of what doing proofs is like in school mathematics today is related to conjecturing and setting up the proof. This is important if you believe, as Lampert (1992) argued, that "conjecturing about…relationships is at the heart of mathematical practice" (p. 308). Related to this is the importance of determining the premises ("givens") and statements to be proved:

> Many people think of geometry in terms of proofs, without stopping to consider the source of the statements that are to be proved.…Insight can be developed most effectively by making such conjectures very freely and then testing them in reference to the postulates and previously proved theorems. (Meserve & Sobel, 1962, p. 230)

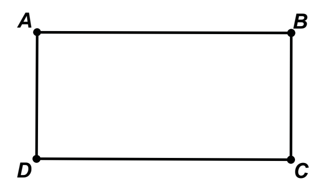
Because we believe that students should play a larger role in the important work of setting up and carefully analyzing proofs, we present problems that are reminiscent of the historical problems described above in that they do not simply provide students with the given hypotheses and ask them to prove particular statements. Rather, we propose nine different problems (presented in no particular order) that illustrate how students may be provided with opportunities to expand their role in the process of proving.

In the first three problems, students are asked to participate in setting up the proof by either providing the premises, the conclusion, and/or the diagram for the proof. In Problem 1, the student is provided with a conjecture (i.e., the diagonals of a rectangle are congruent) and a corresponding diagram and asked to write the "Given" and the "Prove" statements. In contrast, in Problem 2, the student is provided with the "Given" and the "Prove" statements but is asked to draw the diagram.

PROBLEM 1: Writing the "Given" and "Prove" from a conjecture

Suppose you conjectured that the diagonals of a rectangle are congruent and drew the diagram below.

Write the "Given" and the "Prove" statements that you would need to use to prove your conjecture.



PROBLEM 2: Drawing a diagram when provided with the "Given" and the "Prove"

Draw a diagram that could be used to prove the following:

Given: Parallelogram PQRS where T is the midpoint of $\overline{PQ}$ *and V is the midpoint of* $\overline{SR}$.

Prove: $\overline{ST} \cong \overline{QV}$

Finally, in Problem 3, when provided with a particular theorem, the student is asked to do all three of these tasks (i.e., write the "Given," the "Prove," and draw the diagram).

PROBLEM 3: Setting up the "Given," the "Prove," and the diagram when provided with the theorem

Determine what you have been given and what you are being asked to prove in the theorem below. Mark a diagram that represents the theorem.

Theorem: If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Problem 4 is similar to the first three in that students are invited to determine the "Given," but this time they are also provided with the statement to be proved as well as the proof of that statement. Students are asked to determine what would have been "Given" in order to develop the proof that is provided. They are then asked to condense those two "Givens" into a single, more concise statement. This exercise asks students to reflect on two different ways that the line segment bisector premise might be handled. Problem 4 is similar to the "fill in" type proofs that we have seen in some textbooks (e.g., Larson et al., 2001; Serra, 2008), except that rather than having students fill in the statements or reasons, they are filling in the premises.

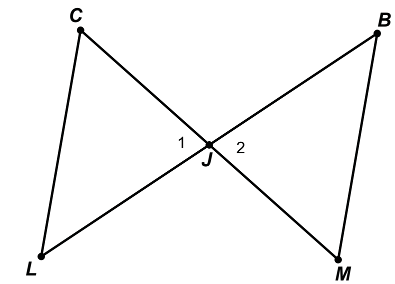
PROBLEM 4: Determining the "Given" from a Flow Proof

1. *Provide the two missing "Given" statements for the proof shown on the next page.*
2. *Write a single statement that could replace these two given statements.*

Given: ____________

____

Prove: $\overline{CL} \cong \overline{MB}$



? (Given)

? (Given)

$\overline{CJ} \cong \overline{MJ}$ (Definition of Midpoint)

$\overline{JL} \cong \overline{JB}$ (Definition of Midpoint)

$\angle 1 \cong \angle 2$ (Intersecting lines form congruent vertical angles)

$\Delta CJL \cong \Delta MJB$ (SAS $\cong$ SAS)

$\overline{CL} \cong \overline{MB}$ (CPCTC)

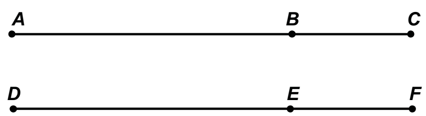
(Adapted from Serra, 2008, p. 239)

Next, in Problem 5, students are asked to draw a conclusion or determine what could be proved when provided with particular "Given" conditions and a corresponding diagram. This type of problem can be a useful scaffold in that it isolates particular geometric ideas such as definitions or postulates of equality, for example.

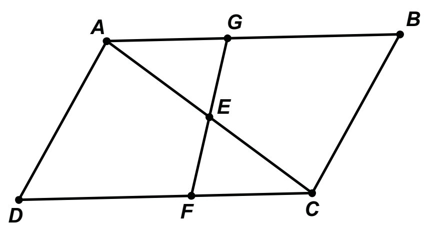
PROBLEM 5: Drawing Conclusions from the "Given"

What conclusions can be drawn from the given information?

Given: $\overleftrightarrow{ABC}$, $\overleftrightarrow{DEF}$
$\overline{AB} \cong \overline{DE}$
$\overline{BC} \cong \overline{EF}$



Given: Quad *ABCD* where $\overline{FG}$ is bisected by diagonal $\overline{AC}$



(Adapted from Lewis, 1978, pp. 135 & 68)

In Problem 6, students are asked to determine what auxiliary line might be drawn in order to construct the proof that two angles are congruent. This is not a common problem posed to students because, typically, teachers either construct the auxiliary lines for their students or a hint is provided in the textbook that helps students determine where this line should be drawn (Herbst & Brach, 2006). We view these first six problems as scaffolds that could eventually allow students to conjecture and set up a proof on their own.

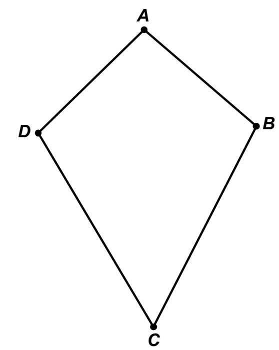
PROBLEM 6: Drawing an auxiliary line.

What auxiliary line might we draw in to construct this proof?

Is it possible to construct the proof without considering an auxiliary line?

Given: Kite *ABCD* with
$\overline{AD} \cong \overline{AB}$ and
$\overline{DC} \cong \overline{BC}$

Prove: $\angle B \cong \angle D$



Problem 7 is unique in the sense that the student is asked what could be proved, but the givens are ambiguous. Leaving the problem more open-ended affords students opportunities to write conjectures. It is expected that the student will consider two different cases corresponding to whether the quadrilateral is concave or convex. In both cases the student could argue that the remaining pair of sides are congruent to each other.

PROBLEM 7: Solving a problem that involves writing a conjecture (i.e., deciding what to prove)

Consider a quadrilateral that has two congruent consecutive segments and two opposite angles congruent. The angle determined by the two congruent sides is not one of the congruent angles. What else could be true about that quadrilateral? What could you prove in this scenario? What are the "Given" statements?

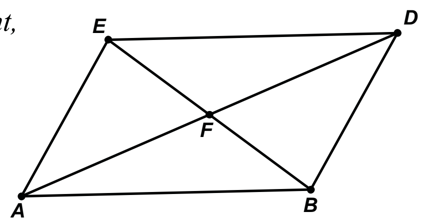
Finally, in Problems 8 and 9, students have the opportunity to take part in analyzing proofs. In Problem 8, a paragraph proof is provided, and students are asked to find the error. In this proof, the corresponding parts that are proved to be congruent are two pairs of angles and one pair of sides. The student author determined that the triangles were congruent by Angle-Side-Angle (ASA) based on the order that these corresponding parts were proved congruent, rather than attending to how these parts are oriented in the triangles. In Problem 9, students are provided with a proof and asked to determine what theorem was proved.

PROBLEM 8: Finding the error in a proof.

In the figure to the right,

$\overleftrightarrow{AB} \parallel \overleftrightarrow{ED}$ *and*

$\overline{AB} \cong \overline{ED}$.



Luis uses this information to prove that $\Delta ABF \cong \Delta DEF$.

Explain why his paragraph proof is incorrect and give a reason why he may have made this error.

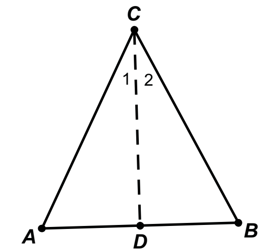
Proof:

It is given that $\overleftrightarrow{AB} \parallel \overleftrightarrow{ED}$ so $\angle DEB \cong \angle ABE$ because parallel lines cut by a transversal form congruent alternate interior angles. It is also given that $\overline{AB} \cong \overline{ED}$. And $\angle AFB \cong DFE \angle$ because they are vertical angles, and vertical angles are congruent. So $\Delta ABF \cong \Delta DEF$ by ASA.

(Adapted from EDC, 2009, p. 122)

PROBLEM 9: Determine the theorem that was proved by the given proof.

Write the theorem that was proved by the proof below.



Statements	Reasons
1. ΔACB with $\overline{CA} \cong \overline{CB}$	1. Given.
2. Let $\overline{CD}$ be the bisector of vertex $\angle ACB$, D being the point at which the bisector intersects $\overline{AB}$.	2. Every angle has one and only one bisector.
3. $\angle 1 \cong \angle 2$	3. A bisector of an angle divides the angle into two congruent angles.
4. $\overline{CD} \cong \overline{CD}$	4. Reflexive property of congruence.
5. $\Delta ACD \cong \Delta BCD$	5. Side-Angle-Side ≅ Side-Angle-Side
6. $\angle A \cong \angle B$	6. Corresponding parts of congruent triangles are congruent.

(Adapted from Keenan & Dressler, 1990, p. 172)

In this section, we proposed nine problems that illustrate how teachers could increase their students' involvement in proving by having them make reasoned mathematical conjectures, use conjectures to set up a proof, and evaluate mathematical proofs by looking for errors and determining what was proved. In the next section, we address the issue of

supporting students in proving by commenting on multiple proof representations.

Proof Representations that Support Developing and Writing Proofs

Representation is one of the five Process Standards which highlight the ways in which students acquire and make use of content knowledge (NCTM, 2000). In particular, various proof forms can be considered as representations of geometric knowledge. Providing students with access to various proof representations is useful because “different representations support different ways of thinking about and manipulating mathematical objects” (NCTM, 2000, p. 360). Although it is important to encourage students to represent their ideas in ways that make sense to them, it is also important that they learn conventional forms of representation to facilitate both their learning of mathematics and their communication of mathematical ideas (NCTM, 2000). The purpose of this section is to highlight four different ways that proofs can be represented in geometry and discuss how these various representations have the potential to facilitate proving.

As pointed out by Anderson (1983), successful attempts at proof generation can be divided into two major episodes—“an episode in which a student attempts to find a plan for the proof and an episode in which the student translates that plan into an actual proof” (p. 193). We refer to these two activities as *developing* and *writing* a proof, respectively. The proof forms that we highlight include proof tree, two-column proof, flow proof, and paragraph proof. Descriptions and examples of each representation can be found in the appendix. In this section we briefly discuss the ways in which these proof representations can support students in developing and writing a proof.

Two-Column Proof

A two-column proof lists the numbered statements in the left column and a reason for each statement in the right column (Larson et al., 2001). The two-column form *requires* that students record the claims that make up their argument (in the statements column) as well as their justifications for these claims (in the reasons column). In this sense, the two-column

form appears to be a rigid representation. This could be intimidating to students. However, students can be flexible when using this representation. For example, they might leave out a reason that they do not know but still move ahead with the rest of the proof; the incomplete form reminds them that they still have something to complete (Weiss, Herbst, & Chen, 2009) However, the consecutively numbered steps of the proof may lead students to believe that the deductive process is more linear than it actually is. The deductive process, in general, hides the struggle and the adventure of doing proofs (Lakatos, 1976).

Paragraph Proof

A paragraph proof describes the logical argument using sentences. This form is more conversational than the other proof forms (Larson et al., 2001). Paragraph proofs are more like ordinary writing and can be less intimidating (EDC, 2009). For this reason, they look more like an explanation than a structured mathematical device (EDC, 2009). However the lack of structure could also be a detriment. In particular, some teachers have concluded that the paragraph form was not appropriate for high school students because students tended to leave out the reasons that justified their statements. As a result, students would often come to invalid conclusions (Cirillo, 2008). Yet, if a goal is to help students develop mathematical literacy, this proof form most closely resembles the representation that a mathematician would use to write up a proof. Another advantage of this form is that when writing a proof by contradiction, the paragraph form seems a more sensible choice than some of the other options (Lewis, 1978).

Proof Trees

The proof tree is an outline for action, where the action is *writing* the proof. Anderson (1983) described the proof tree as follows:

> The student must either try to search forward from the givens trying to find some set of paths that converge satisfactorily on the statement to be proven, or [s/he] must try to search backward from the statement to be

> proven, trying to find some set of dependencies that lead back to the givens. (p. 194)

In other words, students might begin by asking themselves, “What would I need to do in order to prove this statement?” Using a proof tree to think through a proof could be a useful scaffold to support students in *developing* a proof. The proof tree could also be a useful tool to scaffold the work of determining what the given premises are or what conclusion can be proved.

Flow Proof

A flow proof uses the same statements and reasons as a two-column proof, but the logical flow connecting the statements is indicated by arrows (Larson et al., 2001) and separated into different “branches.” The flow proof helps students to brainstorm, working through the most difficult parts of solving a proof: (1) understanding the working information—analyzing the given and the diagram—and (2) knowing what additional information is needed to solve the proof—analyzing what is being proved (Brandell, 1994). The flow proof form also allows students to see how different subarguments can come together to make the overarching argument (i.e., the “prove” statement). A disadvantage to this proof form might be that students are not required to supply reasons that justify their statements in the way that the “Reasons” column of the two-column proof forces them to do. For that reason, however, it allows students to focus on organizing the argument and thus could be particularly useful toward developing a proof.

The Teacher’s Role in Managing Proof Activity

Through his work, Stylianides (2007) concluded that teachers must play an active role in managing their students' proving activity by making judgments on whether certain arguments qualify as proofs and selecting from a repertoire of courses of action in designing instructional interventions to advance students' mathematical resources related to proof. One way that we can see teachers playing this active role is through their use and allowance of various representations of proof. More specifically, acceptance of these various representations

of proof allows teachers and their students to focus more on the argument rather than its form. This can be done through the side-by-side presentation of a flow proof and a two-column proof that presented the same argument, as we observed in one secondary classroom. In this case, the teacher emphasized to his students that he was not as concerned with the form of the proof as he was with the presentation of valid reasoning (Cirillo, 2008).

Lampert (1992) noted:

> Classroom discourse in ‘authentic mathematics’ has to bounce back and forth between being authentic (that is, meaningful and important) to the immediate participants and being authentic in its reflection of a wider mathematical culture. The teacher needs to live in both worlds in a sense belonging to neither but being an ambassador from one to the other. (p. 310)

If teachers can be flexible in their thinking about the form that proofs might take, while at the same time concerning themselves with the content of the argument, then students may have more success in learning to prove. Furthermore, the examples we provide suggest that teachers could also enrich students’ proving experiences by creating opportunities for students to do more than producing an argument that links the givens and the prove. The experiences of students can be more authentic if they have opportunities to hypothesize the premises needed to prove a conclusion, to make deductions from a set of premises so as to find an unanticipated conclusion, and so forth. This affords students opportunities to learn about proof as a mathematical process and participate in mathematics in ways that are truer to the discipline.

Conclusion

Various stakeholders in mathematics education have called for reasoning and proof to play a more significant role in the mathematics classroom. There is some evidence that these recommendations have been taken seriously by mathematics educators and textbook developers. In this paper, however, we argue that if we are truly to realize the goals of these standards, we must pose problems to our students that allow them a

greater role in proving. We presented problems that asked students to write the premises, write the statements to be proved, as well as construct the diagrams. We suggest that students should be provided with opportunities to make reasoned conjectures and evaluate mathematical arguments and proofs. Last, we suggest that teachers promote and allow various types of reasoning and methods of proof. We believe that this is important because adherence to a specific proof format may elevate focus on form over function. A focus on form potentially obstructs the creative mix of reasoning habits and ultimately hinders students' chances of successfully understanding the mathematical consequences of the arguments.

As Lakatos (1976) described using the dialectic of proofs and refutations, mathematicians do not just prove statements given to them, they also use proof to come up with those statements. Teaching practices that involve students in solving problems, conjecturing, writing conditional statements to prove, and then explaining and verifying their conjectures can provide students with more authentic opportunities to engage in mathematics.

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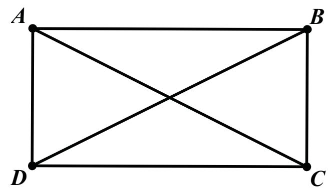
APPENDIX

Proof Representations

THEOREM: If a parallelogram is a rectangle, then the diagonals are congruent.

Given: Rectangle $ABCD$ with diagonals $\overline{AC}$ and $\overline{BD}$.

Prove: $\overline{AC} \cong \overline{BD}$



A **two-column proof** lists the numbered statements in the left column and a reason for each statement in the right column.

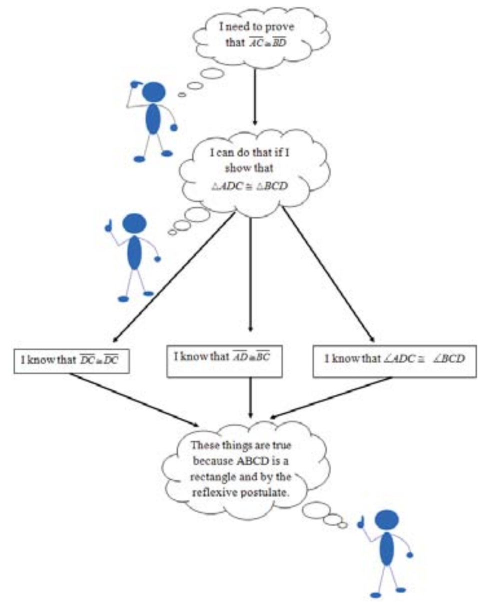
Statements	Reasons
1. Rectangle $ABCD$ with diagonals $\overline{AC}$ and $\overline{BD}$	1. Given
2. $\overline{AD} \cong \overline{BC}$	2. Opposite sides of a rectangle are congruent.
3. $\overline{DC} \cong \overline{DC}$	3. Reflexive Postulate
4. $\angle ADC$ and $\angle BCD$ are right angles.	4. All angles of a rectangle are right angles.
5. $\angle ADC \cong \angle BCD$	5. All right angles are congruent.
6. $\Delta ADC \cong \Delta BCD$	6. Side-Angle-Side ≅ Side-Angle-S
7. $\overline{AC} \cong \overline{BD}$	7. Corresponding Parts of Congrue Triangles are Congruent (CPCTC)

A **paragraph proof** describes the logical argument with sentences. It is more conversational than a two-column proof.

Since ABCD is a rectangle with diagonals $\overline{AC}$ and, $\overline{BD}$ then $\overline{AD} \cong \overline{BC}$ because opposite sides of a rectangle are congruent. By the reflexive postulate $\overline{DC} \cong \overline{DC}$. Since all angles in a rectangle are right angles, then $\angle ADC$ and $\angle BCD$ are right angles. Thus, $\angle ADC \cong \angle BCD$. By Side-Angle-Side, $\Delta ADC \cong \Delta BCD$. Thus, $\overline{AC} \cong \overline{BD}$.

A **proof tree** is an outline or plan of action that specifies a set of geometric rules that allows students to get from the givens of the problem, through intermediate levels of statements, to the to-be-proven statement.

(Adapted from Anderson, 1983)



A **flow proof** uses the same statements and reasons as a two-column proof, but the logical flow connecting the statements is indicated by arrows. Depending on whether it is the plan or the proof itself, students may or may not choose to write the reasons beneath the statements.